MARKING SCHEME

1. (a) Solution

P(A∩B) = 0.7 and P(A∩B) = 0.2

From P(AUB) = (P(A) + P (B) – P(A∩B)

0.7 = P(A) + P(B) - 0.2

P(A) + P(B) = 0.9 …………………………..(i) (1 Mark)

But

P(A) + P(A') = 1 …………………………….(ii)

P(B) + P(B') = 1 …………………………….. (iii) (1 Mark)

Add equations (i) and (ii)

P(A) + P(B) + P(A') + P(B') = 2 ……………. (iv)

Substitute equation (i) into equation (iv)

0.9 + P(A') + P(B)' = 2

P(A') + P(B′) =1.1.

(b) Given that p(x) = 0.5 – rx in the interval 0<x<4

(i) r =?

From $\int_0^4 p(x)dx = 1$ (0.5 Mark)

From $\int_0^4 (0.5 - rx)dx =$ 0.5(4-16 r) = 1 (1 Mark)

The value of r = $\frac{1}{8}$

(ii) E (x) =?

From E (x) = $\int_0^4 xP(x)\,dx$

$$E(x) = \int_0^4 x\left(\frac{1}{2} - \frac{1}{8}x\right)dx = \left[\frac{1}{4}x^2 - \frac{1}{24}x^3\right]_0^4 = \frac{4}{3}$$

The expectation of x is $\frac{4}{3}$ (1 Mark)

(i) Var (x) =?

From Var(x) = $E(x^2) - [E(x)]^2$

Also $E(x^2) = \int_0^4 x^2 p(x)dx$

4 4 4

$E(x^2) = \int_0 x^2 \left(\frac{1}{2} - \frac{1}{8}\right)dx = \int_0 \left(\frac{1}{2}x^2 - \frac{1}{8}x^3\right)dx = \left[\frac{1}{6}x^3 - \frac{1}{32}x^4\right]_0 = \frac{8}{3}$ (1Mark)

$Var(x) = \frac{8}{3} - (\frac{3}{4})^2 = \frac{8}{9}$

Variance of x is $\frac{8}{9}$ (1Mark)

(ii) P(1<x<2) =?

From $P(1<x<) = \int_1^2 p(x)dx$ (0.5Mark)

$\int_1^2 \left(\frac{1}{2} - \frac{1}{8}x\right)dx = \left[\frac{1}{2}x - \frac{1}{16}x^2\right]_1^2 = \frac{5}{16}$

$P(1<x<2) = \frac{5}{16}$ (1 Mark)

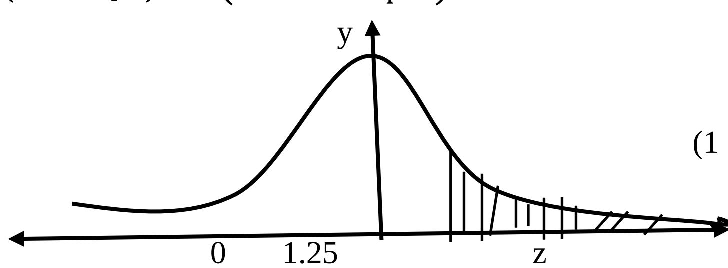
(c) Given, Mean = 20

Standard deviation = 4

But, $Z = \frac{X-\mu}{\delta} = \frac{X-20}{4}$ (0.5Mark)

(i) P(x>25) =?

$P(x>25) = p\left(z = \frac{x-20}{4}\right) = p\left(z >= \frac{25-20}{4}\right) = p\ (z>1.25)$ (0.5 Mark)



(1 Mark)

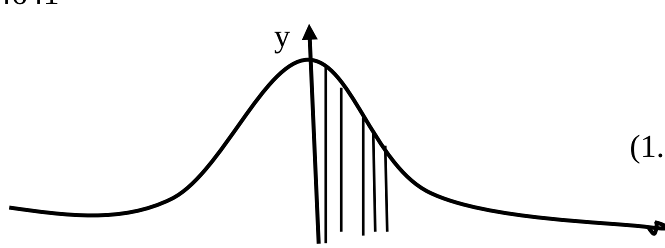
P(x>25) = 0.1057

(ii) d = ? Given that p(20<x<d) = 0.4641

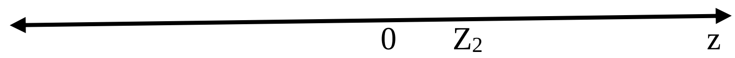
$P\left(\frac{20-20}{4} < z < \frac{d-20}{4}\right) = 0.4641$

$P\left(0 < z < \frac{d-20}{4}\right) = 0.4641$ let $Z_2 = \frac{d-20}{4}$ (0.5Mark)

P(0<z<z2) = 0.4641



(1.5 Mark)



by using z – scoresheet, $Z_2 = 1.8$

but $Z_2 = 1.8 = \frac{d - 20}{4}$

$d = 27.2$ (1 Mark)

2. (a) (i) Sentence is a basic unit of language that express a complete thought.

Example: Mbina is the most talented boy in the class.

While

Statement is a group of words which can be judged to be true or false.

Example: Mbina is the most talented boy. (1 Mark)

(ii) Use a truth table as follows

P	q	p→q	q→p	p ↔ q	(p →q)∧(q → p)
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F		T	T	T

(2 Marks)

From the table above the column for p ↔q is the same as the column (p→q) ∧ (q →p).

Hence, the two sentence are equivalent.

(b) Converse; "If they cancel school, then it rains" (1 Mark)

Contrapositive; "If they do not cancel school, then it does not rain." (1 Mark)

Inverse "If it does not rain, then they do not cancel school." (1 Mark)

(c) Proceed as follow

[(p → ~q) ∧ (q∨r) ∧ p] → r ………………………………………… Given

~ [(~pv~q) ∧ (q v r) ∧ p] vr …………………………………….. by definition

~ [(~pv ~q) ∧p ∧(qvr)] vr ………………………..………………… commutative law

~ [(~p ∧ p)v (~q ∧p) ∧(qvr)]vr ………………………………….. distributive law

~ [(Fv(~ q ∧p) ∧ (qvr)]vr …………………..………………….. complement law

~ [(~q ∧ p) ∧ (qvr)]vr ………………………………………………… identify law

~ (~q ∧p) ∨~ (qvr) vr …………………………………………….... Demorgan's law

(qv ~p)v(~q∧~r)vr …………………………………………… Demorgan's law

(qv ~p)v[(~qvr) ∧ (~rvr)]…………………………………… Distributive law

(qv ~p)v[(~qvr) ∧T]……………..………………………………… Compliment law

(qv ~p)v[(~qvr) ………………………….………………………… Identity law

(qv ~p)v[(~qvr) ………………………….………………………… Complement law

Tv (~pvr) …………………………………………………………… Complement law

T ……………………………………………………………….. Identity law

Hence the proposition is tautology (4Marks)

(d) Consider the rows containing F on the last column.

[(~x v ~ y v ~ z) ∧ (~x v ~yvz)] ∧ (xv ~y v ~ z) …………………… Given

[(~x v ~ y) v (~ z ∧ z)] ∧ (x v ~y v ~ z) ……………………..……… Distributive law

[(~x v ~ y) v F] ∧ (x v~–y v) ~ z) ……………………..……………….. Complement law

(~x v~y) ∧ (x v ~y v~z) ……………………………………………... Identity law

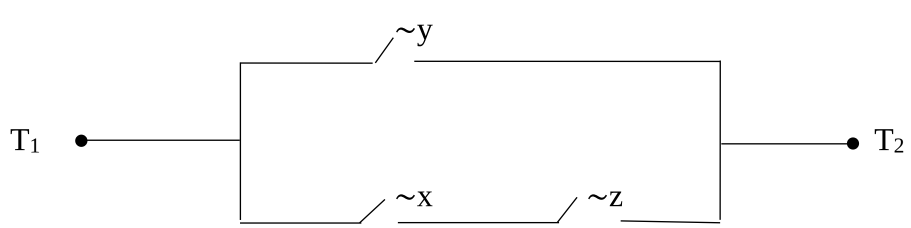
(~y v ~ x) ∧ (~y v x v~z) ……………………………………..Commutative law

~y v [~x∧ (x v ~z)] …………………….………………………… Distributive law

~y v [(~x ∧ x)v(~ x ∧~z)] …………………………………………. Distributive law

~ y v [F v (~x ∧ ~z)] ……………………………………………Complement law

~ y v (~x∧~z)) ……………………………………………Identity Law



(2Marks)

3. (a) (i) Proceed as follow

Required a vector of 8 units in a direction of vector 5i –j + 2k

Let the required vector = "b" in the direction â = 5i – j +n 2k

ƀ =8a ……………………………………………………….. (i) (0.5 Mark)

$\hat{a} = \frac{\hat{a}}{|\hat{a}|} = \frac{5i-j+2k}{\sqrt{5^2+(-1)^2+2^2}} = \frac{5i-j+2k}{\sqrt{30}}$ …………………………… (ii) (1 Mark)

substitute equation (ii) into equation (i)

$b = 8\left(= \frac{5i-j+2k}{\sqrt{30}}\right) = \frac{8}{\sqrt{30}}(5i - j + 2k)$

A vector is $\frac{8}{\sqrt{30}}(5i - j + 2k)$ (1Mark)

(ii) Proceed as follow

Let $\underline{a} = i - 2j + 3k$, $|\underline{a}| = \sqrt{14}$

$\underline{b} = 3i - 2j + 3k$, $|\underline{b}| = \sqrt{14}$

From $\underline{a}.\underline{b} = |\underline{a}||\underline{b}| \cos\theta$

$\theta = \cos^{-1}\left(\frac{\hat{a}.b}{|a||b|}\right)$ ……………………………………….. (i) (1 Mark)

$\underline{a}.\underline{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 10$ ………………………………(ii) (1 Mark)

$|\underline{a}||\underline{b}| = (\sqrt{14})(\sqrt{14}) = 14$ ………………………... (iii)

$\theta = b \cos^{-1}\left(\frac{10}{14}\right) = 44^0 24'$

The angle between the vectors is $44^0 24 \approx 44.42^0$ **(1 Mark)**

(b) Proceed as follows

$\underline{a} = 3i + j - 2k$, $b = I - 3j - k$ and $m:n = 1:3$

Internal division = $\left(\frac{m}{m+n}\right)\underline{b} + \left(\frac{m}{m+n}\right)\underline{a}$

Internal division = $\left(\frac{1}{1+3}\right)(i - 3j - k) + \left(\frac{3}{1+3}\right)(3i + j - 2k)$

Internal division = $\frac{1}{4}(i - 3j - k) + \frac{3}{4}(3i + j - 2k)$

Internal division = $\frac{1}{4}(i - 3j - k + 9i + 3j - 6k)$

Internal division = $\frac{1}{4}(10i - 7k)$

A position vector is $\frac{10}{4}i - \frac{7}{4}k$ (2 Marks)

(c) (i) λ =? when $\underline{a}$ and $\underline{b}$ are perpendicular.

From $\underline{a}.\underline{b} = \underline{|a||b|}$ Cosθ

a.b = $\underline{|a||b|}$ Cosθ 90^0 (1Mark)

a.b = 0

a.b = = $\begin{pmatrix} 2 \\ 4 \\ \lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = 0$

2 +8 + 7λ = 0

$\boldsymbol{\lambda = -\frac{10}{7}}$ (1 Mark)

(ii) λ = ? when $\underline{a}$ and $\underline{b}$ are collinear.

From $\underline{a}.\underline{b} = \underline{|a||b|}$ Cosθ

$\underline{a}.\underline{b} = \underline{|a||b|}$ Cosθ

$\underline{a}.\underline{b} = \underline{|a||b|}$ ………………………………………………….. (i) (1 Mark)

a.b = = $\begin{pmatrix} 2 \\ 4 \\ \lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = 10 + 7\lambda$ ………………………………….. (ii)

$\underline{|a||b|} = \left(\sqrt{20 + \lambda^2}(\sqrt{54}\right)$

$\underline{|a||b|} = \left(\sqrt{54\ (20 + \lambda^2}\right)$ ……………………………………….. (iii)

Substitute equation (ii) and (iii) into equation (i)

$10 + 7\lambda = \sqrt{54(20 + \lambda^2)}$

$(10 + 7\lambda)^2 = 54\ (20 + \lambda^2)$

$100 + 140\lambda + 49\lambda^2 = 1080 + 54\ \lambda^2$

$5\ \lambda^2 - 140 + 980 = 0$

λ = 14. (1 Mark)

(d) Proceed

Let f_m = 6N

= (1, -2,2)

$|f| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

$F = f_m \left(\frac{f}{|f|}\right) = 6\left(\frac{i-2j+2k}{3}\right) = 2(i-2j + 2k)$

F = 2i – 4j + 4k ………………………………………………….. (i) (1 Mark)

The distance from a = (0,1,2) to b (-1,3,-2)

d = b – a = (-1,3,-2) – (0,1,2) = (-1,2,-4)

d = (-1,2, -4) ……………………………………………………… (ii) (1 Mark)

From, work done = F.d

Work done = $\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$ = -2-8 – 16 = $|-26|$ = 26

The work done on moving a particle is 26 units. (1 Mark)

4. (a) Required to solve for z

Given that $z^3 = 1 + i\sqrt{3}$ express in polar form

$|z| = r = 2$, Arg (z) = tan-1 $\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Now, $z^3 = 2\left(Cos\ \frac{\pi}{3} + \frac{\pi}{3}\right)$

From $z_{k+1} = r^{\frac{1}{n}}\left[Cos\left(\frac{\theta+2\pi k}{n}\right) + \ isin\ \left(\frac{\theta+2\pi k}{n}\right)\right]$

From $z_{k+1} = r^{1x}\left[Cos\left(\frac{\frac{\pi}{3}+2\pi k}{3}\right) + \ isin\ \left(\frac{\frac{\pi}{3}+2\pi k}{3}\right)\right]$

From $z_{k+1} = \sqrt[3]{2}\left[Cos\left(\frac{\pi+6\pi k}{9}\right) + \ i\ sin\ \left(\frac{\pi+6\pi k}{9}\right)\right]$ where k = 0,1,2

When k =0;$Z_1 = \sqrt[3]{2}\left[Cos\left(\frac{\pi}{9}\right) + \ i\ sin\ \left(\frac{\pi}{9}\right)\right]$ (1 Mark)

When k =1; $Z_2 = \sqrt[3]{2}\left[Cos\left(\frac{7\pi}{9}\right) + \ i\ sin\ \left(\frac{7\pi}{9}\right)\right]$ (1 Mark)

When k =2;$Z_3 = \sqrt[3]{2}\left[Cos\left(\frac{13\pi}{9}\right) + \ i\ sin\ \left(\frac{13\pi}{9}\right)\right]$ (1 Mark)

(b) Required to show that; tan $3\theta = \frac{3t-t^2}{t-3t^2}$ where t = tanθ

Consider tan$3\theta = \frac{Sin3\theta}{Cos3\theta}$ ……………………………..………………….. (i)

First express z = Cos3θ and Z = sinθ into powered form

$(Cos\theta + isin\theta)^3 = Cos3\theta + isin3\theta$

$Cos3\theta + isin3\theta = (Cos\theta + isin\theta)^3$

$Cos3\theta + isin3\theta = Cos^3\theta + 3iCos^2\theta sin\theta + 3i^2Cos\theta sin^2\theta + i^3sin^3\theta$ but $i^2 = -1$, $i^3 = -i$

$\text{Cos}3\theta + i\sin 3\theta = \text{Cos}^3\theta + 3i\text{Cos}^2\theta\sin\theta - 3\text{Cos}\theta\sin^2\theta - i\sin^3\theta$

$\text{Cos}3\theta + i\sin 3\theta = (\text{Cos}^3\theta - 3\text{Cos}\theta\text{Sin}^2\theta) + i(3\text{Cos}^2\theta\text{Sin}\theta - \sin^3\theta)$ ……………………. (ii)

From Equation (ii) equate real part

$\text{Cos}3\theta = \text{Cos}^3\theta - 3\text{Cos}\theta\text{Sin}^2\theta$

$\text{Cos}^3\theta = \text{Cos}^3\theta - 3\text{Cos}\theta(1 - \text{Cos}^3\theta)$

$\text{Cos}3\theta = \text{Cos}^3\theta - 3\text{Cos}\theta + 3\text{Cos}^3\theta$

$\text{Cos}3\theta = 4\ \text{Cos}^2\theta - 3\text{Cos}\theta$ ………...............………………….. (iii) (1 Mark)

From equation (ii) equate imaginary part

$\text{Sin}3\theta = 3\text{Cos}^2\theta\sin\theta - \sin^3\theta$

$\text{Sin}3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$\text{Sin}3\theta = 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$\text{Sin}3\theta = 3\sin\theta - 4\sin^3\theta$ …………………………………………………. (iv) (1 Mark)

Substitute equations (iii) and (iv) into equation (i)

$\text{Tan}\ 3\theta = \frac{\text{Sin}3\theta}{\text{Cos}3\theta} = \frac{3\text{Sin}3\theta - 4\ \text{Sin}^3\theta}{4\ \text{Cos}^3\theta - 3\text{Cos}\theta}$

$\text{Tan}\ 3\theta = \frac{3\text{Sin}3\theta - 4\ \text{Sin}^3\theta}{4\ \text{Cos}^3\theta - 3\text{Cos}\theta}$

Divide by $\text{Cos}^3\ \theta$ to each term on right hand side

$\text{Tan}\ 3\theta = \frac{3\tan\theta\sec^2\theta - 4\ \tan^3\theta}{4 - 3\ \text{Sec}^2\theta}$ but $\text{Sec}^2\ \theta = 1 + \tan^2\ \theta$

$\text{Tan}\ 3\theta = \frac{3\tan\theta\left(1 + \tan^2\theta\right) - 4\ \tan^3\theta}{4 - 3\ (1 + \tan^2\theta)} = \frac{3\tan\theta + 3\ \tan^3\theta - 4\ \tan^3\theta}{4 - 3 - 3\tan^2\theta}$

$\text{Tan}3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ but $\tan\theta = t$

$\text{Tan}3\theta = \frac{3t - t^3}{1 - 3t^2}$ hence shown. (1 Mark)

Also given, $t^3 - 3t^2 - 3t + 1 = 0$

$\frac{3t - t^3\theta}{t - 3t^2} = 1$

Comparing with; $\tan 3\theta\ \frac{3t - t^3}{t - 3t^2}$ (0.5 Mark)

$\tan 3\theta = 1$

$3\theta = \tan^{-1}(1) = 45^0$

From general formula of tangent; $3\theta = 180^0 n + 45^0$

$\theta = 15^0 (4n + 1)$

$\theta = 15^0, 75^0, 135^0, 195^0, \ldots\ldots\ldots\ldots$ (0.5 Mark)

but; $t = \tan\theta$

$t_1 = 0.2679$, $t_2 = 3.7321$ and $t_3 = -1$ (1 Mark)

(c) Given tan z = 3

But tan = -itanh(iz)

-itanh(iz) =3, tanh (iz) = 3i

$\text{Tanh}(iz)\ \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = 3i$ (1 Mark)

$e^{iz} - e^{-iz} = 3i(e^{iz} + 3ie^{-iz})$

$e^{iz} - e^{-iz} = 3ie^{iz} + e^{-iz}$

$(1 - 3i)\, e^{iz} - (1 + 3i)e^{-iz} = 0$

$(1 - 3i)\, e^{2iz} - (1 + 3i) = 0$

$e^{2iz} = \frac{1+3i}{1-3i}$

$e^{2iz} = \left(\frac{1+3i}{1-3i}\right)\left(\frac{1+3i}{1+3i}\right) = \frac{-4}{5} + \frac{3}{5}i$

$2iz = \ln\left(-\frac{4}{5} + \frac{3}{5}i\right)$

$z = \frac{1}{2i}\ln\left(-\frac{4}{5} + \frac{3}{5}i\right)$

$\therefore z = -\frac{i}{2}\ln\left(-\frac{4}{5} + \frac{3}{5}i\right)$ (2 Marks)

(d) Proceed as follow;

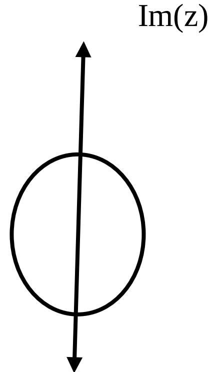
Given that |z| = 3 but z = x + iy

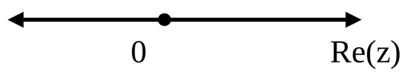
$|x + iy| = 3$

$\sqrt{x^2 + y^2} = 3$

$x^2 + y^2 = 3^2$

Sketch





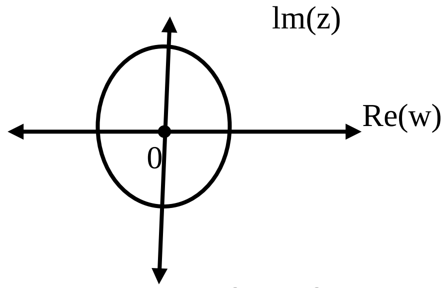
(2 Marks)

Under transformation $|w| = 2\,|z|$ but $|z| = 3$ and w = u + iv

$|u + iv| \quad = 2(3)$

$\sqrt{u^2 + v^2} = 6$

$u^2 + v^2 = 6^2$



(2 Marks)

The image of the circle is $u^2 + v^2 = 6^2$ with centre at origin and radius 6 units.

SECTION B (40 Marks)

Student should any TWO question from this section.

5. (a) (i) Consider L.H.S

$$\sqrt{\frac{1-\text{Sin}\theta}{1+\text{Sin}\theta}} = \sqrt{\frac{(1-\text{Sin}\theta)(1-\text{Sin}\theta}{(1+\text{Sin}\theta((1-\text{Sin}\theta)}} = \sqrt{\frac{(1-\text{Sin}\theta)^2}{1-\text{Sin}^2\theta}} = \sqrt{\frac{(1-\text{Sin}\theta)^2}{\text{Cos}^2\theta}} = \frac{1-\text{Sin}\theta}{\text{Cos}\theta} = \frac{1}{\text{Cos}\theta} - \frac{\text{Sin}\theta}{\text{Cos}\theta}$$

$\sqrt{\frac{1-\text{Sin}\theta}{1+\text{Sin}\theta}}$ **= Secθ – tanθ, hence shown** (3 Marks)

(ii) Required to eliminate θ

x = sinθ + Cosθ ……………………………………………. (i)

y = Sinθ – Cosθ ……………………………………………. (ii)

add equation (i) and (ii)

x + y = 2Sinθ

$\frac{x+y}{2}$ = sinθ …………………………………………………. (iii)

Subtract equations (i) and (ii)

x - y = 2Cosθ

$\frac{x-y}{2}$ = sinθ …………………………………………………. (iv)

Subtract equation (iii) and (iv) into $Cos^2θ + sin^2$

$\left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2 = 1$

∴ $(x-y)^2 + (x + y)^2 = 4$

(b) Required the max, and min, value of a function.

Consider $\frac{1}{5Cosx+12sinx-10} = \frac{1}{RCos(x-\alpha)-10}$

Express 5cosx + 12sinx = Rcos (x - α)

5cosx + 12sinx = R(cosxcosα + sinxsinα)

5cosx + 12sinx = Rcosxcosα + sinxsinα

Equate

5 = Rcosα ……………………………………………………………………. (i)

12 = Rsinα ……………………………………………………………………. (ii)

Square equations (i) and (ii) then add

$5^2 + 12^2 = R^2Cos^2\alpha + R^2Sin^2\alpha$

169 = R2 ($Cos^2\alpha$+ $Sin^2\alpha$)

$R^2 = 169$

R =13 (1Mark)

5Cosx + 12Sinx = 13 Cos(x-α)

Therefore $\frac{1}{5Cosx+12sinx-10} = \frac{1}{13Cos(x-\alpha)-10}$

At maximum Cos(x-α) =1

$\frac{1}{5Cosx+12sinx-10} = \frac{1}{13-10} = = \frac{1}{3}$ (1 Mark)

At minimum Cos(x-α) =-1

$\frac{1}{5Cosx+12sinx-10} = \frac{1}{-13-10} = \frac{-1}{23}$ (1Mark)

∴ The maximum value is $\frac{1}{3}$ and the minimum value is $\frac{-1}{23}$

(c) Given A, B and C as angles of a triangle

Sin2A + Sin2B + Sin2C = 4SinASinBSinC

Consider LHS

Sin2A + Sin2B + Sin2C = Sin(A+B) Cos (A –B) + 2SinCcosC ……………. (i)

From A + B + C = π, A +B = π – C

Sin(A+B) = Sin (π – C) (1Mark)

Sin(A+B) = Sin C …………………………………………………………(ii)

Substitute equation (ii) into equation (i)

Si2A + Sin2B+Sin2C = 2SinCcos (A –B) + 2SinCcos C

Si2A + Sin2B+Sin2C = 2SinC [Cos (A –B) + Cos C] …………………….……….(iii)

Also apply cos into A + B = π – C

Cos(A+B) = Cos(π – C)

Cos(A+B) = -CosC ………………………………………………………………. (iv)

Substitute equation (iv) into equation (iii)

Si2A + Sin2B+Sin2C = 2SinC [cos (A –B) - Cos (A+B)]

Si2A + Sin2B+Sin2C = 2SinC [-2SinASin(-B)]

∴ **Si2A + Sin2B+Sin2C = 4SinASinBSinC** (2Marks)

(d) Required to solve for θ

Sin3 θ - Sin θ = 0

3Sinθ – 4Sin^{3}θ – Sinθ = 0

2Sinθ – 4Sin^{3}θ = 0

Sinθ (1 – 2Sin^{2}θ) = 0

Sinθ=0 or Sinθ = $\pm\frac{\sqrt{2}}{2}$ (2 Marks)

Consider Sinθ = 0

θ=Sin^{-1}0 = 0^0, (α = 0^0)

From θ = πn + (-1)^{n}α where n = 0,1,2

θ = 180^{0}n

$\theta = 0, 180^0$ (1 Mark)

Also Consider Sinθ = $\frac{\sqrt{2}}{2}$

$\theta = \text{Sin-1}\ \frac{\sqrt{2}}{2} = 45^0, (\alpha = 45^0)$

From $\theta = \pi n + (-1)^n\alpha$ where n = 0,1,2

$\theta = 180^0 n - (-1)^n\ 45^0$

$\theta = 45^0, 135$ (1 Mark)

Also consider Sinθ = $\frac{\sqrt{2}}{2}$

$\theta = \text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -45^0, (\alpha = -45^0)$

From $\theta = \pi n + (-1)\ )^n\ \alpha$ where n = 0,1,2…..

$\theta = 180^0 n + (-1)^n\ (-45^0)$

$\theta = 225^0, 315^0$ (1Mark)

6. (a) (i) Given; $\frac{1}{x^3-1}$

$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$ ………………………………………………..…………… (i)

$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$1 = A\ (x^2 + x + 1) + (x - 1)\ (Bx + C)$

$1 = A\ (x^2 + x + 1) + Bx^2 + (C – B)\ x - C$

Equate

$0 = A + B$ ………………………………………………………………… (ii)

$0 = A – B + C$ …………………………………………………………….. (iii)

$1 = A – C$ …………………………………………………………………. (iv)

From equation (iv) C = A – 1 Substitute into equation (iii)

$0 = A – B + A – 1$

$2A – B = 1$ …………………………………………………………… (v)

Solve equations (ii) and (v) simultaneously

$A + B = 0$

$2A - B = 1$

$A = \frac{1}{3}$, $B = \frac{-1}{3}$, $C = A - 1 = -\frac{2}{3}$

$\therefore \frac{1}{X^3-1} = \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$ (2 Marks)

(ii) Assume α,β and γ be the roots of a cubic equation.

Sum of roots = $\alpha + \beta + \gamma = 6$ …………………………………… (i)

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$6^2 = 14 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$2(\alpha\beta + \alpha\gamma + \beta\gamma) = 22$

Sum of product of roots = $\alpha\beta + \alpha\gamma + \beta\gamma = 11$…………………… (ii) (1.5 Marks)

$(\alpha + \beta + \gamma)^3 = (\alpha^3 + \beta^3 + \gamma^3 + 6\alpha\beta\gamma + 3\alpha\beta(\alpha + \beta) + 3\alpha\gamma(\alpha + \gamma) + 3\beta\gamma(\beta + \gamma)$

$6^3 = (36 + 6\beta\gamma + 3\alpha\beta(6-\gamma) + 3\alpha\gamma(6-\beta) + 3\beta\gamma(6-\alpha)$

$216 = 36 + 6\alpha\beta\gamma + 18(\alpha\beta + \alpha\gamma + \beta\gamma) - 9\alpha\beta\gamma$

$180 = 18(11)\ 3\alpha\beta\gamma$

$\alpha\beta\gamma = 6$

Product of roots = $\alpha\beta\gamma = 6$ ……………………………………….. (iii) (1.5Marks)

Now recall the cubic equation;

X^3 – (sum of roots) x^2 + (Sum of product of root) x – (product of roots) = 0

$X^3 - 6x^2 + 11x - 6 = 0$ (0.5 Mark)

$\alpha = 3, \beta = 1$ and $\gamma = 2$ **(1.5 Marks)**

(b) Required to show that; $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n + 1)(2n + 1)$

Consider the series below

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + ... + n^2$

In sigma notation the above series can be written as

$1^2 + 2^2 + 3^2 + 4^2 + 4^2 + ... + n^2 = \sum^{n} r^2$ (1 Mark)

Now consider $\sum_{r=1}^{n}(r + 1)^3 - r^3$

When r = 1, $2^3 - 1^3$

When r = 2, $3^3 - 2^3$

When r = 3, $4^3 - 3^3$

When r = n – 2, $(n-1)^3 - (n-2)^3$

When r = n - 1 $(n^3 - (n-1)^3$

When r = n, $(n+1)^3 - n^3$

$$\sum_{n=1}^{n} (r+1)^3 - r^3 = (n+1)^3 - 1^3$$

$$\sum_{r=1}^{n} (r^3 + 3r^2 + 3r + 1 - r^2) = (n+1)^3 - 1^2$$

$$\sum_{r=1}^{n} (3r^2 + 3r + 1) = (n+1)^3 - 1$$

$$\sum_{r=1}^{n} (3r^2 + \sum_{r=1}^{n} 3r + \sum_{r=1}^{n} 1 = (n+1)^3 - 1$$

$$3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = (n+1)^3 - 1$$

$$3\sum_{r=1}^{n} r^2 + 3\left(\frac{n}{2}(n+1)\right) + n = (n+1)^3 - 1$$

$$3\sum_{r=1}^{n} r^2 = (n+1)^3 - 1 - \frac{3n}{2}(n+1) - n$$

$$3\sum_{r=1}^{n} r^2 = (n+1)^3 - (\frac{3n}{2}(n+1) - (n+1)$$

$$3\sum^{n} r^2 = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{2}$$

$$3\sum^{n} r^2 = \left(\frac{n+1}{2}\right)(2(n+1)^2 - 3n - 2)$$

$$3\sum_{r=1}^{n} r^2 = \left(\frac{n+1}{2}\right)(2(n^{2]0-[7i}+1)$$

$$\mathbf{3\sum_{r=1}^{n} r^2 = \left(\frac{n}{6}\right)(n+1)(2n+1)}$$ (1 Mark)

Now for, $\sum_{r=1}^{n}(r-3)^2$

$$3\sum_{r=1}^{n} (r-3)^2 = \sum_{r=1}^{n} (r^2 - 6r + 9) = \sum^{n} r^2 - 6\sum_{r=1}^{n} r + \sum 9$$

$$3\sum_{r=1}^{n}(r-3)^2 = \frac{n}{6}(n+1)(2n+1) - 6\left(\frac{n}{2}(n+1)\right) + 9n$$

$$3\sum_{r=1}^{n} r^2 = (r-3)^2 = \frac{n}{6}(n+1)(n+1) - 3n(n+1) + 9n$$

$$3\sum_{r=1}^{n} r^2 = (r-3)^2 = \frac{n}{6}(n+1)(2n+1) - 3n^2 + 6n = \frac{n}{6}((n+1)(2n+1) - 18n + 36)$$

$$\therefore\ 3\sum_{r=1}^{n} r^2 = (r-3)^2 = \frac{n}{6}(2n^2 - 15n + 37)$$ (1 Mark)

(c) Proceed as follows;

Consider $\frac{2x^2-7x-4}{3x^2-14x+11} - 2 > 0$

$$\frac{-4x^2+21x-26}{3x^2-14x+11} > 0$$

$$\frac{4x^2-21x+26}{3x^2-14x+11} < 0$$

$$\frac{(4x-13)(x-2)}{(3x-11)(x-1)} < 0$$

Boundaries x = 1, x = 2, x = $\frac{13}{4}$, x = $\frac{11}{3}$ (2 Marks)

	X < 1	1 < x < 2	$2 < x < \frac{13}{4}$	$\frac{13}{4} < x < \frac{11}{3}$	$x > \frac{11}{3}$
4x – 13	- ve	- ve	- ve	+ ve	+ ve
X -2	- ve	- ve	+ ve	+ ve	+ ve
(4x – 13)(x – 2)	+ ve	+ ve	- ve	+ ve	+ ve
(3x – 11)	- ve	- ve	- ve	- ve	+ ve
(x -1)	- ve	+ ve	+ ve	+ ve	+ ve
(3x – 11) (x -1)	+ ve	- ve	- ve	- ve	+ ve
(4x – 13) (x -2)	+ ve	- ve	+ ve	- ve	+ ve

(3x – 11) (x -1)					

∴ $1 < x < 2$ and $\frac{13}{4} < x < \frac{11}{3}$ (1 Mark)

(d) Proceed as follows

Given proposition $\frac{d}{dx}(x^n) = nx^{n-1}$

Step1: We need to prove if it is true for n =1,2

when n =1, $\frac{d}{dx}(x^1) = 1x^{1-1}$

1 = 1 it is true for n = 1 ………………………………………… (i) (1 Mark)

when n = 2, $\frac{d}{dx}(x^2) = 2x^{2-1}$

2x = 2x it true for n =2 ……………………………………….. (ii)

Step2: Assume it is true for n = k

$\frac{d}{dx}(x^k) = k^{\,k-1}$……………………………………………(iii) (1Mark)

Step 3: We need to prove if it is true for n = k +1

$\frac{d}{dx}(x^{k+1}) = (k+1)x^k$

Consider L.H.S and then prove if it is equal to R.H.S

$\frac{d}{dx}(x^{k-1}) = \frac{d}{dx}(xx^k) = x^k \frac{d}{dx}(x) + \frac{d}{dx}(x^k)$

$\frac{d}{dx}(x^{k+1}) = x^k + \frac{d}{dx}(x^k)$ but $\frac{d}{dx}(x^k)=kx^{k-1}$ from equation (iii)

$\frac{d}{dx}(x^{k+1}) = x^k + x\left[kx^{k-1}\right] = x^k + kx^k = (1+k)x^k = (k+1)\ x^{k+1-1}$

$\frac{d}{dx}(x^{k+1}) = (k+1)x^{(k+1)-1}$ but n = k + 1

∴ $\frac{d}{dx}(x^n) = nx^{n-1}$ (1 Mark)

7. (a) Required to form a D.E

$y = x^2 + Ae^{2x} + Be^{3x}$ …………………………….……………… (i)

$y' = 2x + 2Ae^{2x} + 3Be^{3x}$ ……………………………………………… (ii)

$y'' = 2 + 4Ae^{2x} + 9Be^{3x}$ …………………………..…………… (iii)

From equation (i)

$y - x2 - Be^{3x} = Ae^{2x}$ …………………………………………… (iv)

Substitute equation (iv) into equation (ii)

$y' = 2x + 2(y - x^2 - Be^{3x}) + 3Be^{3x}$

$y' - 2x - 2y + 2x^2 = Be^{3x}$ ……………………………………… (v) (0.5 Mark)

Substitute equation (iv) into equation (iii)

$y'' = 2 + 4(y - x^2 - Be^{3x}) + 9Be^{3x}$

$y'' - 2 - 4y + 4x^2 = 5Be^{3x}$ ……………………………………….…. (vi) (0.5 Mark)

Substitute equation (v) into equation (vi)

$y'' - 2 - 4y + 4x^2 = 5(y' - 2x - 2y + 2x^2)$ ………….……….………..…. (vi) (0.5 Mark)

$\therefore y'' - 5y' + 6y = 6x^2 - 10x + 2$ ………………………………………………… (2Marks)

(b) Solving a D.E

$\frac{dy}{dx} = \frac{x-y+2}{x+y}$ …………………………………………………………… (i)

Let $x = X + h, y = Y + k$ and $\frac{dy}{dx} = \frac{dY}{dX}$ ……………………….……..… (ii)

Substitute equation (ii) into equation (i)

$\frac{dY}{dY} = \frac{(X+h)-(Y+k)+2}{(X+h)+(Y+k)}$

$\frac{dY}{dX} = \frac{(X-Y+h-k+2)}{X+Y+h+k}$ ……………………………………………………… (iii)

let and solve

$h - k + 2 = 0$

$h + k = 0$

$h = -1, k = 1$

Then equation (iii) change to

$\frac{dY}{dX} = \frac{X-Y}{X+Y}$ (This is homogeneous D.E) …………………… (iv) (1Mark)

Let $Y = UX$, $\frac{dY}{dX} = U + X\frac{dU}{Dx}$ ………………..……….. (v) (1 Mark)

Substitute equation (v) into equation (iv)

$U + X\frac{dU}{dX} = \frac{X-UX}{X+UX}$

$U + X\frac{dU}{dX} = \frac{1-U}{1+U}$

$X\frac{dU}{dX} = \frac{1-U}{1+U} - U$

$X\frac{dU}{dX} = \frac{1-2U-U^2}{1+U}$

$\int \frac{1+U}{I-2U-U^2} \, dU = \int \frac{1}{X} \, dx$

$-\frac{1}{2} \ln(1 - 2U - U^2) = \ln AX$

$\ln\left(1 - 2\frac{Y}{X} - \frac{Y^2}{X^2}\right) = \ln (AX)^{-1}$

$\left(\frac{X^2-2XY-Y^2}{X^2}\right) = \frac{C}{X^2}$

$X^2 - 2XY - Y^2 = C$ ………………………………………….(vi) (1Mark)

But X = x – h, Y = y - k and h= -1, k = 1

i.e X =x + 1, Y = y -1 ……………………………………… (vii)

substitute equation (vii) into equation (vi)

$(x+1)^2 - 2(x+1)(y-1) - (y-1)^2 = C$ (1 Mark)

$\therefore\ x^2 - y^2 + 4x - 2xy + 2 = C$

Given; y = 1 when x = 0

on substituting the values of x and y into the equation; C = - 1 (0.5 Mark)

$\mathbf{x^2 - y^2 + 4x - 2xy + 3 = 0}$

(c) Solving $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$.

$y'' - 2y' + y = 0$

Write it auxiliary Quadratic Equation (A.Q.E)

$m^2 - 2m + 1 = 0$

m=1

let $\alpha = \beta = p = 1$

From general solution D.E with two distinct roots

$y = (Ax + B)e^{px}$

$\therefore y_{cf} = (Ax + B)e^x$ (Complimentary solution) (I Mark)

Also consider Particular Integral Part, since e^x and xe^x is contained then,

$y = kx^2e^x$ ……………………………………………… (ii)

$y' = kx^2e^x+2kxe^x$ …………………………………….. (iii) (1 Mark)

$y'' = kx^2e^x + 4kxe^x + 2ke^x$ ……………………………… (iv)

Substitute equations (ii), (iii) and (iv) into equation (i)

$(kx^2e^x + 4kxe^x + 2ke^x) – 2(kx^2e^x + 2kxe^x) + 2ke^x= 6e^x$

$2ke^x = 6e^x$

Equate L.H.S and R.H.S to obtain the value k

k=3

$\therefore y_{pI}= 3x^2e^x$ (Particular Integral Solution) (1 Mark)

Therefore, $e_y = y_{cf} + y_{p.I}$ (General Solution)

$\therefore$ **$y = (Ax +B)e^X + \underline{3x^2e^X}$** (2 Marks)

(d) Proceed as follows

$6\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$ ………………………………….. (i)

$z = \frac{dy}{dt}, \frac{d^2z}{dt^2} = \frac{d^2y}{dt^2}$ (1 Mark)

$6\frac{dz}{dt} + z = 0$ (This separable D.E)

$6\int \frac{1}{Z} dz = -\int dt$

6lnz = -t + A

$6\ln z = -\frac{t}{6} + B$

$Z = e^{\left(-\frac{t}{6}+B\right)} = e^{\left(-\frac{t}{6}\right)}e^B = Ce^{\left(-\frac{t}{6}\right)}$

$\frac{dy}{dt} = Ce^{\left(-\frac{t}{6}\right)}$ (1 Mark)

$\int dy = C\int e^{\frac{-t}{6}} dt$

$y = -6Ce^{\left(\frac{-t}{6}\right)} + D$

when y = 63, t = 0

63 = -6C + D …………………………………………. (i)

When y = 36, t = 6ln4

72 = -3C + 2D …………………………………………. (ii)

Solve equation (i) and equation (ii) simultaneously

$$\begin{cases} 63 = -6C + D \\ 72 = -3C + 2D \end{cases}$$

C = -6, D = 27

Then from $y = -6\, e^{\left(\frac{-t}{6}\right)} + D$

$\therefore y = 36e^{\left(\frac{-t}{6}\right)} + 27$

$y = 36e^{\left(\frac{-t}{6}\right)} + 27$

$\frac{dy}{dt} = -6e^{\left(\frac{-t}{6}\right)}$ but $\frac{dy}{dt}$ is rate of cooling must be negative

$\frac{dy}{dt} = 6e^{\left(\frac{-t}{6}\right)}$ Given that $\frac{dy}{dt} < 1$

$6e^{\left(\frac{-t}{6}\right)} = \frac{dy}{dt} < 1$ (1Mark)

$6e^{\left(\frac{-t}{6}\right)} = < 1$

$t < -6 \ln \frac{1}{6}$

$t < 10.75$

The rate of cooling of the body will have fallen below one degree per minute after 10 minutes.

8. (a) proceed as follows;

$4x^2 + 25y^2 - 24x + 50y - 39 = 0$

$4x^2 - 24x + 25y^2 + 50y = 39$

$4(x^2 - 6x) + 25(y^2 + 2y) = 39$ (by completing the square)

$4(x - 3)^2 + 25(y+1)^2 = 100$

$\frac{(x-3)^2}{25} + \frac{(y+1)^2}{4} = 1$ compare with $\frac{(x-h)^2}{a^2} + \frac{(y+k)^2}{b^2}$ (1 Mark)

$a^2 = 25$, $a = 5$

$b^2 = 4$, $b = 2$

$h = 3$, $k = -1$

$\therefore$ The centre of an ellipse is (3, -1) (1Mark)

Since a > b then $a^2(1 - e^2) = b^2$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$

∴ Eccentricity is $\frac{\sqrt{21}}{5}$ (1 Mark)

∴ Foci are $(\pm ae+h,k) = (\pm\sqrt{21} + 3, -1)$ (1 Mark)

∴ Equation of directrices are $x = \pm\frac{a}{e} + h = \pm\frac{25}{\sqrt{21}} + 3$ (1 Mark)

(b) Required, to show a condition for a line $lx + my = n$ to touch a hyperbola $xy = C^2$

Substitute a line $lx + my = n$ into hyperbola $xy = C^2$

$x\left(\frac{(n-lx)}{m}\right) = C^2$ (1 Mark)

$nx - lx^2 = mc^2$

$lx^2 - nx + mc^2 = 0$

Check if it satisfy perfect square (i.e condition for tangency, $b^2 = 4ac$)

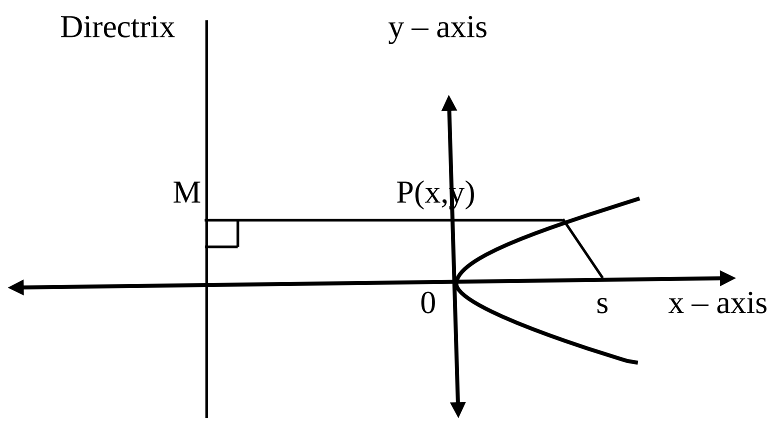
$b^2 = n^2$

$4ac = 4mlc^2$

But $n^2 = 4mlc^2$ (2 Marks)

∴ The given line is a tangent to the rectangular hyperbola

(c) (i) Parabola is a conic sections whose eccentricity is one (e = 1) (1 Mark)

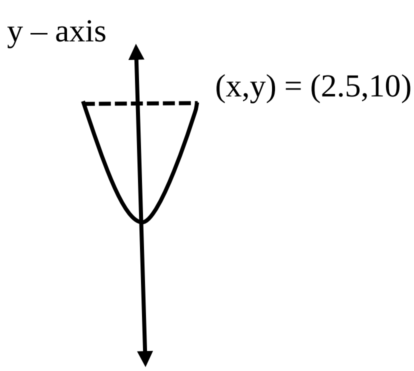


(1.5 Marks)

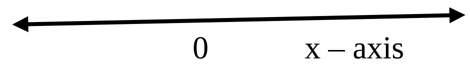
Where; $e = \frac{SP}{MP}$ (0.5 Mark)

(ii) Proceed as follows;

Consider the figure below



(1Mark)



From $r^2 = 4by$ at $(x,y) = (2.5,10)$

$(2.5)^2 = 4b(10)$, $b = \frac{5}{32}$

Then $x^2 = \frac{5}{8}y$ (1Mark)

At $y = 2$, $x = \frac{\sqrt{5}}{2}$

$\therefore$ Wide of an arch $= 2x = 2\left(\frac{\sqrt{5}}{2}\right) = \sqrt{5}$ (2Marks)

(e) Proceed as follows;

Table of value $r = 3\cos 2\theta$

0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
3	1.5	-1.5	-3	-1.5	1.5	3	1.5	-15	-3	-1.5	1.5	3

(2Marks)

The curve of $r = 3\cos 2\theta$